

Bounding the Feedback Vertex Number of Digraphs in Terms of Vertex Degrees

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Abstract

The Turán bound [17] is a famous result in graph theory, which relates the independence number of an undirected graph to its edge density. Also the Caro–Wei inequality [4, 18], which gives a more refined bound in terms of the vertex degree sequence of a graph, might be regarded today as a classical result. We show how these statements can be generalized to directed graphs, thus yielding a bound on directed feedback vertex number in terms of vertex outdegrees and in terms of average outdegree, respectively.

Keywords: directed feedback vertex number, Caro–Wei inequality, feedback set, acyclic set, induced DAG

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1. Introduction

Not only in discrete mathematics, generalizing existing concepts and proofs has always been a guiding theme for research. The great mathematician Henri Poincaré even considered this as the *leitmotiv* of all mathematics.¹

In particular, many results from graph theory were generalized to weighted graphs, digraphs, or hypergraphs. Sometimes, providing such generalizations is an easy exercise; in other cases, the main difficulty lies in formulating the “right generalization” of the original theorem. An additional obstacle is imposed if the result we intend to generalize allows for several proofs or equivalent reformulations. Then there are many roads to potential generalizations to explore, and selecting the most promising one can be difficult. However, once the proper generalizations of the used notions are found, the more general proof often runs very much along the same lines.

As we shall see, one such example is the Turán bound [17], which gives the number of edges that a graph of order n can have when forbidding k -cliques as subgraphs. It allows for many different proofs and equivalent reformulations, see [1]. A dual version of Turán’s bound, regarding the size of independent sets, was refined by Caro [4] and Wei [18]. Their result has subsequently been generalized, by replacing the independent sets with less restricted induced subgraphs [3],

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¹Poincaré’s original phrasing was more poetic: “La mathématique est l’art de donner le même nom à des choses différentes.” [14, p. 29].

respectively by replacing the concept of a graph with more general notions, namely weighted graphs [16] and hypergraphs [5]. Here we complement these efforts by providing a generalization of the Caro–Wei bound to the case of digraphs. From an algorithmic perspective, the new result gauges a simple greedy heuristic for the minimum directed feedback vertex set problem. In this way, the main result of this paper yields a formalized counterpart to the intuition that the minimum (directed) feedback vertex number of sparse digraphs cannot be “overly large”.

2. Preliminaries

We assume the reader is familiar with basic notions in the theory of digraphs, as contained in textbooks such as [10]. Nevertheless, we briefly recall the most important notions in the following. A *digraph* $D = (V, A)$ consists of a finite set, referred to as the set $V(G) = V$ of *vertices*, and of an irreflexive binary relation on $V(G)$, referred to as the set of *arcs* $A(G) = A \subset V \times V$. The cardinality of the vertex set is referred to as the *order* of D . In the special case where the arc relation of a digraph is symmetric, we also speak of an (*undirected*) *graph*. For a vertex v in a digraph D , define its *out-neighborhood* as $N^+(v) = \{u \in V \mid (u, v) \in A, u \neq v\}$, and its *out-degree* as $d^+(v) = |N^+(v)|$. In-neighborhood and in-degree are defined analogously, and denoted by $N^-(v)$ and $d^-(v)$, respectively. The *degree* $d(v)$ of v is then defined as $d(v) = |N^-(v) \cup N^+(v)|$, and the *total degree* of v is defined as $|N^-(v)| + |N^+(v)|$. We note that our definition of vertex degree agrees (on undirected graphs) with the standard usage of this notion in the theory of undirected graphs, see e.g. [7]. For a subset of vertices $U \subseteq V$ of the digraph $D = (V, A)$, the *subdigraph induced by U* is the digraph $(U, A|_{U \times U})$ obtained by reducing the vertex set to U and by restricting the arc set to the relation induced by A on U . If a digraph H can be obtained in this way by appropriate restriction of the vertex set of the digraph D , we say H is an *induced subdigraph* of D . A *simple path* in a digraph is a sequence of $k \geq 1$ arcs $(v_1, w_1)(v_2, w_2) \cdots (v_k, w_k)$, such that for all $1 \leq i < k$ holds $w_i = v_{i+1}$ and all start-vertices v_i are distinct. If furthermore $w_k = v_1$, we speak of a *cycle*. In particular, notice that each pair of opposite arcs $(v, w)(w, v)$ in a digraph amounts to a cycle. This convention is commonly used in the theory of digraphs, compare [10].

A digraph containing no cycles is called *acyclic*, or a *directed acyclic graph (DAG)*. For a vertex subset U of a digraph D , if the subdigraph induced by U is acyclic, then we call U an *acyclic set*. In particular, if $D[U]$ contains no arcs at all, then U is called an *independent set*. The maximum cardinality among all independent sets in D is called the *independence number* of D . Turán proved the following bound on the independence number of undirected graphs:

Theorem 1. *Let $D = (V, A)$ be an undirected graph of order n and of average degree \bar{d} . Then D contains an independent set of size at least $(\bar{d} + 1)^{-1} \cdot n$.*

Caro [4], and, independently, Wei [18] proved the following refined bound:

Theorem 2. *Let $D = (V, A)$ be an undirected graph of order n . Then D contains an independent set of size at least $\sum_{v \in V} (d(v) + 1)^{-1}$.*

A set F of vertices in a digraph $D = (V, A)$ is called a *feedback vertex set* if $V \setminus F$ is an acyclic set. The *feedback vertex number* $\tau_0(D)$ of D is defined as the minimum cardinality among all feedback vertex sets for D . A simple observation is that for a digraph D of order n , the cardinality of a maximum acyclic set equals $n - \tau_0$.

3. Directed Feedback Vertex Sets and Vertex Degrees

Quite recently, several new algorithms were devised for exactly solving the minimum directed feedback vertex set problem [6, 15]. But all known exact algorithms for this problem share the undesirable feature that their worst-case running time is exponential—in the order n of the input graph, or at least in the size of the feedback vertex number τ_0 . This is not surprising as the problem has been known for a long time to be **NP**-complete, see [8].

Here, we consider the following simple greedy heuristic for finding a large acyclic set, and hence a small feedback vertex set, in a digraph D . We call the algorithm *Min-Greedy*, in accordance with a homonymous greedy heuristic on undirected graphs for finding a large independent set, compare [9, 11].

Starting with $D_1 = D$, we inductively define a sequence of digraphs D_i , $i \geq 1$, by first choosing a vertex v_i , such that v_i has minimum outdegree in D_i , and then deleting v_i , along with its out-neighborhood in D_i , to obtain the digraph D_{i+1} . We proceed in doing so until the vertex set of D_i is empty, and remember the vertices v_i selected in each turn. These vertices form the set $S = \{v_1, v_2, \dots, v_r\}$, which is the result finally returned by the procedure.

Before we analyze the quality of the above heuristic, we shall first prove its soundness.

Lemma 3. *Let D be a digraph. Then the set S returned by Min-Greedy on input D is an acyclic set in D .*

Proof. Using the notions from the description of the algorithm, it suffices to show that for all $v_j, v_k \in S$, the condition $j < k$ implies that the digraph D has no arc (v_j, v_k) . This claim implies that along every simple path in $D[S] = D[\{v_1, v_2, \dots, v_r\}]$, the vertex indices must appear in decreasing order, thus ruling out the possibility of a cycle in $D[S]$.

To prove the claim, observe first that for all $k > j$, D_k is an induced subdigraph of D_{j+1} . Thus starting with $D_{j+1} = D_j - (\{v_j\} \cup N^+(v_j))$, no vertex in the out-neighborhood of v_j is present in any of the subsequent digraphs. But v_k is selected from G_k , hence is present in G_k and cannot be in the out-neighborhood of v_j . \square

Observe that the proof of Lemma 3 does not depend on the choice of a vertex of minimum degree in D_i for v_i —the algorithm is sound if we choose any vertex in D_i for v_i . Now we are ready to state our main result.

Theorem 4. *Let $D = (V, A)$ be a digraph of order n . Then Min-Greedy always finds an acyclic set in D of size at least $\sum_{v \in V} (d^+(v) + 1)^{-1}$.*

Proof. Using the notation from the algorithm, let v_i be the selected vertex of minimum outdegree in D_i . Then for all vertices $w \in N_{D_i}^+(v_i) \cup \{v_i\}$ holds

$$d_D^+(w) + 1 \geq d_{D_i}^+(w) + 1 \geq d_{D_i}^+(v_i) + 1 = |N_{D_i}^+(v_i) \cup \{v_i\}|.$$

Thus,

$$\begin{aligned} \sum_{w \in N_{D_i}^+(v_i) \cup \{v_i\}} (d_D^+(w) + 1)^{-1} &\leq \sum_{w \in N_{D_i}^+(v_i) \cup \{v_i\}} (d_{D_i}^+(w) + 1)^{-1} \\ &\leq \sum_{w \in N_{D_i}^+(v_i) \cup \{v_i\}} (|N_{D_i}^+(v_i) \cup \{v_i\}|)^{-1} = 1. \end{aligned}$$

On the other hand, since the algorithm partitions the vertex set of D into a disjoint union of subsets as

$$V(D) = \bigcup_{i=1}^{|S|} (N_{D_i}^+(v_i) \cup \{v_i\}),$$

we have

$$\sum_{v \in V} (d_D^+(v) + 1)^{-1} = \sum_{i=1}^{|S|} \sum_{w \in N_{D_i}^+(v_i) \cup \{v_i\}} (d_D^+(w) + 1)^{-1} \leq \sum_{i=1}^{|S|} 1 = |S|,$$

as desired. \square

Just like the Caro–Wei bound [4, 18] for the independence number of undirected graphs implies the Turán bound [17] by the inequality of arithmetic and harmonic means, we have the following simple bound on the size of a maximum acyclic set, and hence, on the directed feedback vertex number, in terms of average outdegree:

Corollary 5. *Let $D = (V, A)$ be a digraph of order n and of average outdegree $\overline{d^+}$. Then*

$$\tau_0(D) \leq n \cdot \left(1 - (\overline{d^+} + 1)^{-1} \right).$$

Proof. We show the equivalent statement that the digraph D has an acyclic set of cardinality at least $n / (\overline{d^+} + 1)$. The bound $\sum_{v \in V} (d^+(v) + 1)^{-1}$ from Theorem 4 looks different at first glance. Nevertheless, it easily implies a bound in terms of average outdegree: recall that the *inequality of the harmonic, geometric and arithmetic mean* (see [2, Chapter 16]) states that the geometric mean of n positive numbers a_1, a_2, \dots, a_n is sandwiched between the harmonic mean and the arithmetic mean of these numbers, that is,

$$\frac{n}{\sum_{i=1}^n a_i^{-1}} \leq \left(\prod_{i=1}^n a_i \right)^{1/n} \leq \sum_{i=1}^n a_i / n.$$

Now choose the a_i to be the vertex degrees in D increased by 1 each. Then the outermost inequality yields $\frac{n}{\sum_{v \in V} (d^+(v) + 1)^{-1}} \leq \sum_{v \in V} (d^+(v) + 1) / n$. A very simple calculation completes the proof. \square

Both the bound from Theorem 4 and the one from Corollary 5 are sharp, as witnessed, for example, by the digraph of order $k \cdot m$ that is obtained as the disjoint union of m many k -cliques.

Notice that we obtain the Caro–Wei bound and the Turán bound, respectively, if we restrict the scope of the above statements to symmetric digraphs: for these, the size of a maximum acyclic set is equal to the independence number, and the outdegree of each vertex is equal to its degree (which in turn is equal to half its total degree).

As a final note, we remark that the Moon–Moser bound on the number of maximal independent sets in undirected graphs [13] does not generalize to an analogous statement about maximal acyclic sets; as a matter of fact, not even tournaments allow for a clear generalization [12]. In the undirected case, the proofs of both the Caro–Wei bound and the Moon–Moser bound can be used to derive Turán’s graph theorem, see [1]. A general theme for further research is to identify those fragments of the theory of undirected graphs that generalize smoothly to the case of digraphs.

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